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EXAMPLES OF NON-UNIQUENESS FOR THE LINEAR PROBLEM OF POTENTIAL FLOW AROUND SEMI-SUBMERGED BODIES†

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The plane Neumann–Kelvin problem, which uses a linear approximation of the theory of waves of small amplitude to describe the steady, vortex-free motion of semi-submerged cylindrical bodies in an ideal, incompressible, heavy liquid with a free surface, is considered. For each fixed value of the free-stream velocity and a convoy of two or more bodies it is shown that the geometry of the bodies can be so chosen that the homogeneous Neumann–Kelvin problem will have a non-trivial solution. A family of potentials is constructed that provide such examples of non-uniqueness. The corresponding configurations can be obtained by choosing some of the streamlines of the solutions as the body contours. Examples are given.

The classical work in this field includes [1, 2], which are devoted to the Neumann–Kelvin problem for a totally submerged body. Subsequent attempts were made to apply that boundary-value problem to the motion of bodies which emerge from the free surface of a liquid (see [3–5], for example). The condition that the velocity vector must be bounded everywhere in the liquid was commonly replaced by the condition that it must be bounded outside a compact set which included the body. This led to singularities of the velocity vector at the points of intersection of the body surface and the free surface, resulting in an open boundary-value problem. Several kinds of additional conditions have been suggested (see [3–5], for example). We will use an extension of the formulation proposed in [3].

1. FORMULATION OF THE PROBLEM

We will describe the boundary-value problem using the example of a single body. The notation is shown in Fig. 1, where W is the cross-section of the region containing the liquid, $F = F_+ \cup F_-$ is the free surface of the liquid, D and S are the cross-section and wetted surface of the cylinder, U is the constant velocity of the body along the abscissa axis, and g is the acceleration due to gravity. It will be assumed that the unilateral tangents to S at the points $P_\pm$ make angles $B_\pm \neq 0, \pi$ with the rays $F_\pm$. The unit normal directed into the region occupied by the liquid is denoted by $\mathbf{n}$.

We will use dimensionless coordinates $x = Xg/U^2$ and $y = Yg/U^2$ below. Then, in a system of coordinates associated with the body, the Neumann–Kelvin problem for finding the velocity potential $u \in H^1_{loc}(W)$, reduced to dimensionless form using the coefficient g/U^3, takes the form

$$\nabla^2 u = 0 \text{ in } W \tag{1.1}$$

$$u_{xx} + u_y = 0 \text{ on } F \setminus \{P_+, P_-\} \tag{1.2}$$

$$\partial u / \partial n = f \text{ on } S \setminus \{P_+, P_-\} \tag{1.3}$$

$$\lim_{x \to +\infty} |\nabla u| = 0 \tag{1.4}$$

$$\sup\{ |\nabla u| : (x, y) \in W \setminus E \} < \infty \tag{1.5}$$

where E is a compact set for which $\overline{D} \subset E$ and $E \cap (F_\pm \setminus \{P_\pm\}) \neq 0$. If $f = \cos(n, x)$, then (1.3) is the impermeability condition.

We must supplement Eqs (1.1)–(1.5) by two conditions, and will use the asymptotic form of the solution near a nodal point established in [4] for this purpose. We introduce polar coordinates $(\rho_\pm, \theta_\pm)$ with

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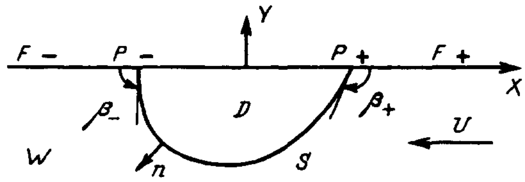

Fig. 1.

pole at the point $P_\pm$ and polar axis in the direction of the ray $F_\pm$. The angle θ_+ (θ_-) will be taken (anti-) clockwise so that $0 \leqslant \theta_\pm \leqslant \beta_\pm$. Then if $S \in C^2$ and $f \in C^1(S)$, as $\rho_\pm \to 0$ we have

$$u = C_\pm + B_\pm \rho_\pm^{\pi/(2\beta_\pm)} \sin(\pi\theta_\pm / (2\beta_\pm)) + A_\pm \rho_\pm \cos(\theta_\pm - \alpha_\pm) + O(\rho_\pm^{\lambda_\pm}) \quad \text{for} \quad \beta_\pm \neq \pi/2$$

$$u = C_\pm + B_\pm[\rho_\pm \log\rho_\pm \sin\theta_\pm + \rho_\pm(\theta_\pm - \pi/2)\cos\theta_\pm] + A_\pm\rho_\pm \cos(\theta_\pm - \alpha_\pm) + O(\rho_\pm^{\lambda_\pm}) \tag{1.6}$$

for $\beta_\pm < \pi/2$.

Here $\alpha_\pm$ and $\lambda_\pm$ are constants, and $1 < \lambda_\pm < 2$ for $\beta_\pm \geqslant \pi/2$ and $\lambda_\pm = 2$ for $\beta_\pm < \pi/2$.

If the submerged part of the body is a semicircle, the formulation given in [3] gives the so-called "least singular solution" with velocity vector bounded at nodal points. In [4] this was applied to an arbitrary contour with $\beta_\pm \geqslant \pi/2$, but it follows from (1.6) that the velocity fields have singularities. Below we will use an appropriate generalization for contours for which $\beta_+ < \pi/2$ or $\beta_- < \pi/2$.

Definition. The potential u is called the "least singular solution" if it is a solution of Eqs (1.1)–(1.5) and satisfies the conditions

$$B_+ = B_- = 0 \tag{1.7}$$

where $B_\pm$ are the coefficients in (1.6).

This formulation is easily extended to the case of an arbitrary number of half-submerged bodies. The results obtained in [4] apply to that case also.

Our aim here is to construct examples of non-uniqueness for problem (1.1)–(1.5), (1.7), or to find potentials which satisfy the homogeneous condition (1.3) and have finite energy. We will use the idea suggested in [6] to construct examples of non-uniqueness for the oscillations of a liquid in which there is a floating body. We place a source and sink of equal intensity on the free surface of the liquid and take certain streamlines of a given potential as the body contours. We will show that these contours can be chosen so that each of the singularities lies inside one of the cylinders thus defined. For this geometry, the proposed potential is a solution of problem (1.1)–(1.5), (1.7) with homogeneous Neumann condition (1.3) on the wetted surface of the bodies. The maximum number of streamlines which can be chosen as body contours for the potential thus constructed depends on the distance between sources. Below we will classify the examples of non-uniqueness as a function of this parameter.

It can be hypothesized that the existence of examples of non-uniqueness is due to the unsteadiness of the flow past the resulting configurations for corresponding values of the velocity.

2. GREEN'S FUNCTION

Green's function $G(x, y; \xi, \eta)$ of the Neumann–Kelvin problem, which satisfies conditions (1.2) and (1.4), the equation

$$\nabla_{x,y} G(x, y; \xi, \eta) = -\delta(x - \xi, y - \eta) \quad \text{for} \quad y < 0, \ \eta < 0$$

and the condition

$$\overline{\lim_{|x+iy|\to\infty}} |\nabla_{x,y} G| < \infty$$

can be written in the form

$$G(z;\zeta) = -(2\pi)^{-1}\,\mathrm{Re}\{\log((z-\zeta)(z-\bar{\zeta})) - 2e^{-i(z-\bar{\zeta})}[\mathrm{Ei}(i(z-\bar{\zeta})) - i\pi]\} \tag{2.1}$$

Here $z = x + iy$, $\zeta = \xi + i\eta$ and $\mathrm{Ei}(z)$ is the integral exponential function.

Consider the case where there is a source on the free surface. We introduce the stream function of the source $H(z;\,\zeta)$, which is the conjugate of Green's function with respect to the argument z. Using formula 8.212.5 of [8], we write

$$H(x,y;\xi,0) = -\pi^{-1}\arg(z-\xi) + \pi^{-1}\,\mathrm{v.p.}\int_0^\infty \frac{e^{ky}\sin k(x-\xi)}{k-1}\,dk - e^{y}\cos(x-\xi) \tag{2.2}$$

where $\arg(z) \in [-\pi, 0]$ for $y \leqslant 0$. The relation

$$\arg(z) = -\frac{\pi}{2} + \int_0^\infty \frac{e^{ky}\sin kx}{k}\,dk, \quad y \leqslant 0$$

which follows from formula 3.941.1 in [8], yields a different representation

$$H(x,y;\xi,0) = -\pi^{-1}\left(\mathrm{v.p.}\int_0^\infty \frac{e^{ky}\sin k(x-\xi)}{k(k-1)}\,dk + \frac{\pi}{2}\right) - e^{y}\cos(x-\xi) \tag{2.3}$$

Using the expansion of $\mathrm{Ei}(z)$ as $z \to 0$ (cf. 5.1.10 in [7]), it can be seen that the function $H(x, y;\,\xi, 0)$ is continuous at the point $(\xi, 0)$.

3. EXAMPLES OF NON-UNIQUENESS

We will consider the family of potentials u_n obtained by placing a source and sink of equal intensity on the free surface at a distance which is a multiple of π, and the corresponding stream functions

$$v_n(x, y) = \pi H(x, y;\, \pi n, 0) - \pi H(x, y;\, -\pi n, 0) \tag{3.1}$$

The latter are even functions of x, so that we can restrict our consideration to the region $x \geqslant 0$.

Note. As $|z| \to \infty$ and $|\zeta| < C < \infty$, we have (see [4], for example)

$$G(z;\zeta) = -\pi^{-1}\log|z| - \vartheta(-x)2e^{y+\eta}\sin(x-\xi) + O\left(|z|^{-1}\right)$$

where ϑ is the Heaviside function. Thus, the functions u_n are so defined that the asymptotic forms at infinity do not contain either a logarithmic or a wave component.

Let $\mathfrak{R}_0$ denote the set of level lines of the function $v_n(x, y)$, the end-points of which lie on the free surface, and let all the streamlines which occur in this set be parametricized using the variable $t \in [0, 1]$. We will also consider $\mathfrak{R}_1 = \{(x(t), y(t)) \in \mathfrak{R}_0\colon \exists (x_i(t), y_i(t)) \in \mathfrak{R}_0,\ [x_i(0), x_i(1)] \subset (x(0), x(1)),\ i = 1, 2,\ (x_1(0), x_1(1)) \cap (x_2(0), x_2(1)) = \varnothing\}$, the set of streamlines which include more than one family of streamlines.

We define $\mathfrak{R} = \mathfrak{R}_0 \backslash \mathfrak{R}_1$. We will introduce the ratio of homotopic equivalence $\rho \subset \mathfrak{R} \times \mathfrak{R}$: the two streamlines $\gamma(t)$ and $\gamma'(t)$, where $t \in [0, 1]$, are homotopic, $(\gamma, \gamma') \in \rho$, if there is a function $\Phi(t, s)$ with the following properties: $\Phi(t, s)$ is continuous with respect to $t \in [0, 1]$ and $s \in [0, 1]$; $\Phi(t, s) \in \mathfrak{R}$ for any $s \in [0, 1]$; $\Phi(t, 0) = \gamma(t)$, $\Phi(t, 1) = \gamma'(t)$. We will prove the following theorem.

Theorem 1. The power of the factor-set $\mathfrak{R}/\rho$ is equal to $2n + 1$.

Let T_1 and T_2 be sets of streamlines which include the sources. It will be shown below that $T_1 \neq \varnothing$ and $T_2 \neq \varnothing$. Let Q_1, Q_2 and T_i $(3 \leqslant i \leqslant 2n + 1)$ denote classes of homotopically equivalent contours such that $Q_1 \supset T_1$ and $Q_2 \supset T_2$. Then by Theorem 1, $\mathfrak{R}/\rho = \{Q_1, Q_2, T_3, \ldots, T_{2n+1}$. We choose certain contours $S = \gamma_1 \cup \gamma_2 \cup \ldots \gamma_N$, where $2 \leqslant N \leqslant 2n + 1$ and $\gamma_i \in T_i$. Obviously, when the given family of lines S is taken as the contours for problem (1.1)–(1.5), (1.7), the corresponding potentials u_n are the solutions of the problem with the homogeneous condition (1.3).

4. REPRESENTATIONS OF THE FUNCTION υ_n

Comparing representations (2.2) and (2.3), for a fixed value of x we have the differential equations $2\partial\upsilon_n\partial y = \upsilon_n - V_n$, which has the solution

$$\upsilon_n(x,y) = e^y\left(\upsilon_n(x,0) + \int_y^0 V_n(x,t)e^{-t}dt\right) \tag{4.1}$$

Here

$$V_n(x, y) = \arg(x + \pi n + iy) - \arg(x - \pi n + iy)$$

and, obviously

$$0 \leqslant V_n(x, y) \leqslant \pi \quad \text{for} \quad y \leqslant 0 \tag{4.2}$$

As we have pointed out, the integrand has a constant sign, and therefore the term in (4.1) represented by the integral is a strictly monotone function of the variable y.

5. THE FUNCTIONS $\upsilon_n(x, 0)$

We will now consider the behaviour of the functions $\upsilon_n(x, 0)$ in the interval $[0, \pi n]$. From (3.1) and (2.2) we have

$$\upsilon_n(x,0) = \pi + \int_0^\infty \frac{\sin k(x-\pi n) - \sin k(x+\pi n)}{k-1}dk$$

Using formulae 3.722.5 and 3.354.1 of [8], we obtain

$$\begin{gathered}\upsilon_n(x, 0) = r(x) + I(x) \\ r(x) = \pi(1 - 2\cos(x - \pi n)), \quad I(x) = I^+(x) + I^-(x) \\ I^\pm(x) = \int_0^\infty \frac{e^{(\pm x - \pi n)k}}{1+k^2}dk\end{gathered} \tag{5.1}$$

We will find bounds for the function $I(x)$. We have

$$I^+(x) \leqslant \int_0^\infty \frac{dk}{1+k^2} = \frac{\pi}{2}, \quad x \in [0, \pi n], \quad I^-(x) < \int_0^\infty e^{-\pi k}dk = \frac{1}{\pi}$$

Combining the resulting inequalities, we obtain

$$I(x) < 2\pi/3 \tag{5.2}$$

For $x \in [0, \pi n]$ we have $\max\{r(x)\} = 3\pi$ and $\min\{r(x)\} = -\pi$. From the last inequality for $I(x)$, we know that the function $\upsilon_n(x, 0)$ changes sign between each adjacent minimum and maximum of the function $r(x)$ and therefore the equation $\upsilon_n(x, 0) = 0$ has at least one root in that interval. This root is unique.

For if there were several such roots, there would have to be a point at which $\partial v_n(x, 0)/\partial x = 0$. We will show that this is impossible.

Let $R = \{x\colon |r(x)| \leqslant 2\pi/3\}$ denote a set in which the functions $r(x)$ and $I(x)$ are commensurate in absolute value. Obviously

$$\min\{|dr(x)/dx| \colon x \in R\} = 2\pi\sin(\arccos(5/6)) = \pi\sqrt{11}/3$$

Then

$$\frac{dI^\pm(x)}{dx} = \pm\int_0^\infty \frac{ke^{(\pm x - \pi n)k}}{1+k^2}dk$$

Since $k \leqslant e^{k-1}$ when $k \geqslant 0$, we have

$$\left|dI^-(x)/dx\right| \leqslant I^+(1)/e < \pi/(2e)$$

If $x \in R$, then $\pi n - x \geqslant \arccos(5/6) > 1/2$. Also, obviously, if $k \geqslant 0$, then $k \leqslant 2e^{k/2-1}$. Thus

$$\left|dI^+(x)/dx\right| \leqslant 2I^+(x+\tfrac{1}{2})/e < \pi/e$$

Combining the bounds thus obtained and the inequality $\| a | - | b \| \geqslant \max\{|a|, |b|\}$ we have

$$\min\left\{|dI(x)/dx| : x \in R\right\} < \pi/e < \min\left\{|dr(x)/dx| : x \in R\right\} = \pi\sqrt{11}/3$$

If $x > \pi n$, from the representation

$$v_n(x,0) = \int_0^\infty \frac{e^{-(x+\pi n)k} - e^{-(x-\pi n)k}}{1+k^2} dk \tag{5.3}$$

which follows from formulae 3.722.5 and 3.354.1 in [8], we conclude that for $x > \pi n$ $v_n(x, 0)$ is a strictly negative, monotonely increasing function and $v_n(x, 0) \to 0$ as $x \to +\infty$.

We note that $v_n(x, y)$ is an even function of x.

This proves the following lemma.

Lemma 1. The function $v_n(x, 0)$ has $2n$ zeros $\xi_1 < \xi_2 < \ldots < \xi_{2n}$. Also $\xi_i \in (-\pi n, \pi n)$ for $i = 1, 2, \ldots, 2n$.

We shall also need the following lemma.

Lemma 2. There are exactly $2n + 1$ local extrema of the function $v_n(x, 0)$, situated at the points $\chi_1 < \chi_2 < \ldots < \chi_{2n+1}$. Also $\chi_1 \in (-\pi n, \xi_1)$, $\chi_{2n+1} \in (\xi_{2n}, \pi n)$ and $\chi_i \in (\xi_{i-1}, \xi_i)$ $(i = 2, 3, \ldots, 2n)$.

Proof. Obviously, $\partial v_n/\partial x \neq 0$ if $|x| > \pi n$. Then in each of the intervals (ξ_i, ξ_{i+1}) $(i = 1, 2, \ldots, 2n)$, there is at least one extremum of the function $v_n(x, 0)$. We will show that there is also at least one extremum in each of the intervals $(-\pi n, \xi_1)$ and $(\xi_{2n}, \pi n)$.

We have shown that $\xi_{2n} < \xi^* = \pi n - \arccos(5/6)$. Consider $x \in (\xi^*, \pi n)$. We will write the equation $\partial v_n(x, 0)/\partial x = 0$ in the form

$$2\pi \sin(\pi n - x) = dI(x)/dx \tag{5.4}$$

The expression on the left-hand side of (5.4) decreases monotonely and vanishes at $x = \pi n$, and the expression on the right-hand side of (5.4) is strictly positive and monotonely increasing. We have

$$dI(x)/dx < \pi/e < 2\pi \sin(\pi n - \xi^*) = \pi\sqrt{11}/3$$

Thus Eq. (5.4) has a root.

Consider the function $\partial^2 v_n(x, 0)/\partial x^2$ for $x \in [0, \pi n]$. Obviously

$$\partial^2 v_n(x,0)/\partial x^2 = p(x) + q(x) - I(x)$$
$$p(x) = 2\pi\cos(x - \pi n), \quad q(x) = (x+\pi n)^{-1} - (x - \pi n)^{-1}$$

(the function $I(x)$ was defined above).

For $x \in [\pi n - \pi/3, \pi n]$, we have

$$p(x) + q(x) > \pi + 3/\pi$$

and, from (5.2), $\partial^2 v_n(x, 0)/\partial x^2 > 0$ for $x \geqslant \pi n - \pi/3$. It follows that the root of Eq. (5.4) corresponds to the minimum of the function $v_n(x, 0)$.

We have thus established that in the interval $(-\pi n, \pi n)$ the function $v_n(x, 0)$ has extrema at m points, where $m \geqslant 2n + 1$.

We will now investigate the zeros of the function $\partial^2 v_n(x, 0)/\partial x^2$. For $x \in [0, \pi n - \pi/3]$, we have

$$q(x) < 4/\pi, \quad |I(x) - q(x)| < 2\pi/3$$

At the same time, for $x \in [0, \pi n]$ we have $\max\{p(x)\} = -\min\{p(x)\} = 2\pi$. Thus, between each minimum and maximum of $p(x)$, the function $\partial^2 v_n(x, 0)/\partial x^2$ changes sign. Let $P = \{x: |p(x)| < 2\pi/3\}$. Obviously

$$\min\{|dp(x)/dx|: x \in P\} = 2\pi \sin\arccos(1/3) = 4\pi\sqrt{2}/3$$

For $x \in [0, \pi n - \pi/3]$ we have $dq(x)/dx = -(x + \pi n)^{-2} + (x - \pi n)^{-2} < 9/\pi^2$. Using the estimate for $|dI(x)/dx|$ obtained above, we arrive at the inequality

$$|dI(x)/dx - dq(x)/dx| \leq \pi/e + 9/\pi^2 < \min\{|dp(x)/dx|: x \in P\} = 4\pi\sqrt{2}/3$$

It follows that there are exactly $2n$ roots of the equation $\partial^2 v_n(x, 0)/\partial x^2 = 0$ in the interval $(-\pi n, \pi n)$. Since there must be a root of the equation $\partial^2 v_n(x, 0)/\partial x^2 = 0$ between every two extrema of the function $v_n(x, 0)$, we have $m = 2n + 1$. This proves the lemma.

6. THE STRUCTURE OF THE LEVEL LINES OF THE FUNCTION $v_n(x, y)$

Note that it follows from the properties of harmonic functions that there cannot be isolated points $v_n(x, y) = c$. Moreover, the streamlines cannot end inside the liquid. A proof of these properties can be found in [6], for example.

We now consider the behaviour of the zero level lines. In view of (4.1), we can find a solution of the equation $v_n(x, y) = 0$ for fixed x^* from the equation

$$v_n(x^*, 0) = -\int_y^0 V_n(x^*, t)e^{-t}dt \tag{6.1}$$

It follows from the definition of the function V_n and inequality (4.2) that the right-hand side of the last equation is a negative, unbounded and monotonely decreasing function of the depth $|y|$. Thus, a solution of Eq. (6.1) exists and is unique if, and only if, $v_n(x^*, 0) \leq 0$.

It follows from (5.3) and Lemma 1 that $v_n(x, 0) \leq 0$ only for $x \in F_i$ ($i = 1, 2, \ldots, n + 1$), where $F_1 = (-\infty, \xi_1]$, $F_{n+1} = [\xi_{2n}, +\infty)$, $F_{j+1} = [\xi_{2j}, \xi_{2j+1}]$ ($j = 1, 2, \ldots, n - 1$). Let $\gamma_0^{(i)}$ denote zero level lines, so that $\gamma_0^{(1)}$. Then the lines $F_i = \mathrm{pr}_x(\gamma_0^{(i)})$ ($i = 1, 2, \ldots, n + 1$) and $\gamma_0^{(n+1)}$ depart to infinity, and the contours $\gamma_0^{(j)}$ ($j = 2, 3, \ldots, n - 1$) are bounded.

Obviously, all the negative level lines starting in the interval F_i lie inside the contour $\gamma_0^{(i)}$ ($1 \leq i \leq n + 1$) and are homotopic to it. This is true by virtue of Lemma 2, which guarantees that no two lines of the same level begin in the interval F_i. Lines of non-zero levels which start in the intervals F_1, F_{n+1} cannot depart to an infinite point (cf. the note in Section 3) and end on the free surface. Note also that since $\xi_1 > -\pi n$ and $\xi_{2n} < \pi n$, there are contours which encompass sources.

We will consider the contours $Re^{i\theta}$, $-\pi \leq \theta \leq 0$ as $R \to \infty$. We use formula (2.1) and the asymptotic representation of the function $\mathrm{Ei}(z)$ for $\mathrm{Re}(z)$ and $|z| \to \infty$

$$\mathrm{Ei}(z) = i\pi\,\mathrm{sign}(\mathrm{Im}(z)) + e^z\left\{\sum_{k=1}^{N} (k-1)!z^{-k} + O\left(|z|^{-N-1}\right)\right\}$$

(the last formula is easily obtained from formulae 5.1.7 and 5.1.51 of [7]). It follows that $v_n(x, y) = -2\pi n(R^{-1}\sin\theta + R^{-2}\cos 2\theta + O(R^{-3})$ and as $x \to -\infty$ ($x \to +\infty$) the line $\gamma_0^{(1)}$ ($\gamma_0^{(n+1)}$) gets closer and closer to the line $y = 1$.

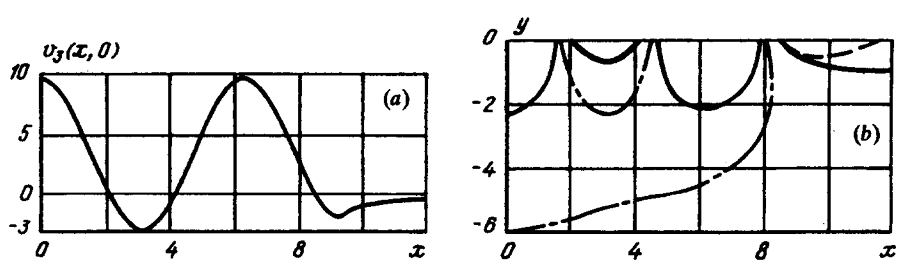

Fig. 2.

We will consider lines of positive level. From (4.1) and (4.2) we obtain the bound

$$v_n(\xi_i, y) = e^y \int_y^0 V_n(\xi_i, t) e^{-t} dt \leqslant \pi e^y (e^{-y} - 1) < \pi, \quad i = 1, 2, \ldots, 2n$$

At the same time, it follows from (5.1) that $\upsilon_n(\chi_{2j}, 0) > 3\pi$ ($j = 1, 2, \ldots, n$). Thus lines which start on the free surface for values of x which belong to one of the intervals forming the set $\{x: \upsilon_n(x, 0) > \pi\}$ must end on the free surface in the same interval, since they cannot intersect the rays $x = \xi_i$ ($i = 1, 2, \ldots, 2n$) and cannot depart to infinity. By Lemma 2, it is impossible to have two lines of the same level starting in one of these intervals.

Thus, we have shown that the maximum number of half-submerged bodies for which the given stream function υ_n gives an example of non-uniqueness of problem (1.1)–(1.5), (1.7), is equal to $2n + 1$ and is the same as the number of local extrema of the function $\upsilon_n(x, 0)$.

We will also point out some features of the streamlines. We will consider one of the contours $\gamma_0^{(m)}$ ($2 \leqslant m \leqslant h$) and a contour γ', $\gamma_0^{(m)} \cap \gamma' = \varnothing$ which is close to it and includes it. It follows from (4.1) that $\upsilon_n(x, y) > 0$ when $(x, y) \in \gamma'$. Then from the analytic properties of $\upsilon_n(x, y)$ it follows that there are contours $\gamma_\delta^{(m)} = \{(x, y): \upsilon_n(x, y) = \delta > 0\}$ for sufficiently small $\delta > 0$, and the quantity $\mathrm{dist}\{\gamma_0^{(m)}, \gamma_\delta^{(m)}\}$ is small. Let $\xi_i(\delta)$ ($i = 1, 2, \ldots, 2n$) denote the roots of the equation $\upsilon_n(x, 0) = \delta$, numbered from left to right. The points $(\xi_{2m}(\delta), 0)$ and $(\xi_{2m+1}(\delta), 0)$ ($m = 1, 2, \ldots, n - 1$) are finite for contours $\gamma_\delta^{(m)}$, each of which includes the corresponding contour of zero level $\gamma_0^{(m)}$. The streamlines which start at points $(\xi_1(\delta), 0)$ and $(\xi_{2n}(\delta), 0)$ cannot depart to infinity and therefore form a single contour (Fig. 2). It is only possible for the streamlines to have this structure if there are $n - 1$ saddle points of the function $\upsilon_n(x, y)$ at which the level lines intersect.

The results of the calculation are shown in Fig. 2. Figure 2(a) shows the function $v_3(x, 0)$ and Fig. 2(b) shows the level lines $v_3(x, y) = c$, where the solid, dashed and dot–dash curves correspond to the values $c = 3.5$; $-1/3$ and 2.2 respectively. The thicker lines correspond to the level $c = 0$. Since the graphs are symmetrical about the coordinate axes, we have shown only those parts which correspond to positive values of x.

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